Dark Solitons for a Generalized Korteweg-de Vries Equation with Time-Dependent Coefficients

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We consider the evolution of long shallow waves in a convecting fluid when the critical Rayleigh number slightly exceeds its critical value within the framework of a perturbed Korteweg-de Vries (KdV) equation. In order to study the wave dynamics of nonlinear pulse propagation in an inhomogeneous KdV media, a generalized form of the considered model with time-dependent coefficients is presented. By means of the solitary wave ansatz method, exact dark soliton solutions are derived under certain parametric conditions. The results show that the soliton parameters (amplitude, inverse width, and velocity) are influenced by the time variation of the dependent model coefficients. The existence of such a soliton solution is the result of the exact balance among nonlinearity, third-order and fourth-order nonlinear dispersions, diffusion, dissipation, and reaction.

Key words: Dark Soliton Solution; Convecting Fluid; Variable-Coefficient KdV Equation; Solitary Wave Ansatz Method.

1. Introduction

Propagation of solitons in nonlinear systems has been the subject of intense research in recent years. This should not be surprising because the soliton approach is universal in different fields of modern physics. The existence of such pulses cover various branches of physics as for example nonlinear optics, plasmas, fluid dynamics, condensed matter physics, and many more. Envelope solitons are stable nonlinear wave packets that preserve their shape when propagating in a nonlinear dispersive medium [1]. It needs to be noted that the formation of this kind of pulses is due to an exact balance between nonlinearity and dispersion effects under specific conditions.

Dark solitons, in contrast to bright solitons, have received a minor attention, in spite of their interesting properties and possible applications in different physical contexts. Regarding to the dynamics of solitons in optical fiber systems for example, it was shown both numerically and analytically that the time jitter in a dark soliton is lower than in the corresponding bright soliton [2,3]. Note that there exist several essential differences between bright and dark solitons [4]. One of them consists of the existence of multiple bound states that can form bright solitons in clear contrast with dark solitons [4].

Solitons are solutions of a special class of partial differential equations (PDEs) that model nonlinear phenomena in physical systems like water waves and light pulses propagation in optical fibers, etc. Well-known PDEs with soliton solutions include the sine-Gordon (sG), cubic nonlinear Schrödinger (NLS), and Korteweg-de Vries (KdV) equations [5]. Importantly, the existence of solitons critically depends on the specific properties of the nonlinear and dispersive terms in the model equations.

The research on the KdV equation attracted the interest of many authors. The KdV equation is the generic model for the study of weakly nonlinear long waves [6]. It arises in physical systems which involve a balance between nonlinearity and dispersion at leading-order [6]. The balance between the nonlinear convection term uu_x and the dispersion effect term u_{xxx} in the spatially one-dimensional KdV equation [7]

$$u_t + a(u^2)_x + u_{xxx} = 0 \tag{1}$$

gives rise to solitons.

If the evolution of steeper waves of shorter wavelength is considered, the higher-order effects become essential and are non-negligible in studying the wave dynamics of nonlinear pulse propagation. Extending the KdV equation to the higher-order KdV equation received much attention, due to the many important applications of different versions of this wave equation. As an interesting example, the following perturbed KdV equation [8,9]

$$u_t+\lambda_1 uu_x+\lambda_2 u_{xxx}+\lambda_3 u_{xxxx} \\ +\lambda_4(uu_x)_x+\lambda_5 u_{xx}=0 \tag{2}$$

is used to study the evolution of long shallow waves in a convecting fluid when the critical Rayleigh number slightly exceeds its critical value. The respective coefficients $\lambda_i, i=1-5$, are given by [8,9]:

$$\begin{aligned}
\lambda_1 &= \frac{3}{2\sigma G}(10+\sigma G),\\
\lambda_2 &= \frac{\sigma\sqrt{G}}{2}\left(\frac{1}{3}+\frac{34}{21}\sigma\right),\\
\lambda_3 &= \frac{682\sigma^2 G+717}{2079}\varepsilon\sigma,\\
\lambda_4 &= \frac{8}{\sqrt{G}}\varepsilon,\\
\lambda_5 &= \frac{R_2}{15}\varepsilon\sigma,
\end{aligned} \tag{3}$$

where σ is the Prandtl number, G the Galileo number, and ε a small parameter such that the excess of the Rayleigh number above its critical value is given by $\varepsilon^2 R_2$ [8,9].

When inhomogeneous systems are considered, nonlinear wave equations with variable coefficients become more realistic than their constant-coefficients counterparts. For example, the generalized KdV equation with t-dependent coefficients [10]

$$u_t+2\beta(t)u+[\alpha(t)+\beta(t)x]\,u_x \\ -3A\gamma(t)uu_x+\gamma(t)u_{xxx}=0 \tag{4}$$

is used to describe the wave dynamics in a varied KdV media.

In 1993, the KdV equation (1) was generalized to the K(m,n) equation

$$u_t+(u^m)_x+(u^n)_{xxx}=0, \\ m>0, 1<n<3, \tag{5}$$

by Rosenau and Hyman [11] to understand the role played by the nonlinear dispersion in the formation of patterns in liquid drops. The compactons [11–13], which are solitons with compact support or strict localization of solitary waves have been found for this class of PDEs.

Ever since the discovery of the K(m,n) equation, much effort has been devoted to finding families of fully nonlinear evolution equations which are generalization forms of the well-known wave equations. Thus, various newly formed nonlinear evolution equations such that the generalized Boussinesq equation (called B(m,n) equation) [14], the coupled Klein-Gordon equations (called CKG(m,n,k)) [15], and the nonlinear Schrödinger equation [16,17], have recently been introduced and studied.

2. Proposed Model

It has been shown in [8] that the evolution equation of surface waves in a convecting fluid is found to obey the perturbed KdV equation (2) that includes diffusion and instability effects, when the critical Rayleigh number slightly exceeds its critical value. It is natural to ask the following question: what kind of KdV equation can be used to study the wave evolution in the case of inhomogeneous convecting fluid? As a matter of fact, this problem is much more general since in realistic physical systems, no media is homogeneous due to long distance of propagation and the existence of some non-uniformity due to many factors as for example variations of the system geometry (diameter fluctuations, etc). It is commonly believed that nonlinear wave equations with variable coefficients are considered to study the wave dynamics in varied systems. In this paper, the perturbed KdV equations (2) will exhibit time-dependent coefficients and also will be generalized to a more general model as follows:

$$(u^l)_t+\lambda_1(t)(u^m)_x+\lambda_2(t)(u^n)_{xxx} \\ +\lambda_3(t)(u^p)_{xxxx}+\lambda_4(t)\left[u^h(u^k)_x\right]_x+\lambda_5(t)(u^q)_{xx} \\ +2\lambda_6(t)u^l+[\lambda_7(t)+\lambda_6(t)x]\,(u^l)_x=0. \tag{6}$$

In (6), the first term is the generalized evolution term, the second term represents the nonlinear convection term, the third and fourth terms, respectively, represent the third-order and fourth-order nonlinear dispersions, while the fifth and sixth terms describe the nonlinear diffusion and dissipation effects, respectively. Also, the time-dependent coefficients $\lambda_i(t)$, with $i=1-7$, are all real valued functions, while l,m,n,p,h,k, and $q\in\mathbb{Z}^+$.

Thus, (6) may be regarded as a combined form of (2) and (4) having time-dependent coefficients. In particular, the case $l = n = p = h = k = q = 1$, $m = 2$, $\lambda_6(t) = \lambda_7(t) = 0$, and $\lambda_i(t)$ constants (with $i = 1-5$) leads to the perturbed KdV equations (2). Moreover, when $l = n = 1, m = 2, \lambda_1(t) = -\frac{3}{2}A\lambda_2(t)$ with A being a constant, and $\lambda_i(t) = 0$ (with $i = 3-5$), (6) reduces to (3). If taking $\lambda_i(t) = 0$ (with $i = 3-5$), (6) can be reduced to a generalized form of the K(m,n) equation having t-dependent coefficients which has recently been considered by Triki and Wazwaz [18].

It is very interesting to note that this generalized KdV equation incorporating various important effects, having time-dependent coefficients and general values of exponents in the existing effects is able to describe the weakly nonlinear long waves dynamics in many physical systems.

Generally, (6) is not integrable. The purpose of this paper is to calculate the exact dark soliton solution for (6) using the solitary wave ansatz method, and find the conditions of their existence for general values of the exponents l,m,n,p,h,k,q, and r and time-dependent model coefficients. It is very interesting to note that the proposed method has been applied successfully to different equations, such as, for example, the NLS equation with power law nonlinearity [19, 20], the K(m,n) equation with t-dependent coefficients [18], and the generalized NLS equation [19]. Further, many other nonlinear wave equations have been recently solved using the solitary wave ansatz method (see for example [14, 19]).

3. Dark Soliton Solutions

In order to construct dark soliton solutions for (6), we use an ansatz solution of the form [17, 18]

$$u(x,t) = A\tanh^s\{\mu(x-vt)\}, \tag{7}$$

where $A = A(t)$, $\mu = \mu(t)$, and $v = v(t)$ are time-dependent coefficients which will be determined as functions of the model coefficients $\lambda_i(t), i = 1-7$. Here A, μ, and v are, respectively, the amplitude, the inverse width, and the velocity of the soliton. The exponent s will be determined as a function of l,m,n,p,h,k, and q.

From ansatz (7), we get

$$\left(u^l\right)_t = lA^{l-1}\frac{\mathrm{d}A}{\mathrm{d}t}\tanh^{sl}\theta + A^l sl\left\{x\frac{\mathrm{d}\mu}{\mathrm{d}t} - \frac{d(t\mu v)}{\mathrm{d}t}\right\} \cdot\left\{\tanh^{sl-1}\theta - \tanh^{sl+1}\theta\right\}, \tag{8}$$

$$(u^m)_x = A^m\mu sm\left\{\tanh^{sm-1}\theta - \tanh^{sm+1}\theta\right\}, \tag{9}$$

$$(u^q)_{xx} = A^q\mu^2 sq\left\{(sq-1)\tanh^{sq-2}\theta + (sq+1)\tanh^{sq+2}\theta - 2sq\tanh^{sq}\right\}, \tag{10}$$

$$\begin{aligned}(u^n)_{xxx} = A^n\mu^3 sn\{&(sn-1)(sn-2)\tanh^{sn-3}\theta\\ &-\left[2s^2n^2 + (sn-1)(sn-2)\right]\tanh^{sn-1}\theta\\ &+\left[2s^2n^2 + (sn+1)(sn+2)\right]\tanh^{sn+1}\theta\\ &-(sn+1)(sn+2)\tanh^{sn+3}\theta\},\end{aligned} \tag{11}$$

$$\begin{aligned}(u^p)_{xxxx} = A^p\mu^4 sp\{&(sp-1)(sp-2)(sp-3)\\ &\cdot\tanh^{sp-4}\theta + (sp+1)(sp+2)(sp+3)\tanh^{sp+4}\theta\\ &-4(sp-1)\left[s^2p^2 - 2sp + 2\right]\tanh^{sp-2}\theta\\ &-4(sp+1)\left[s^2p^2 + 2sp + 2\right]\tanh^{sp+2}\theta\\ &+2sp\left[3s^2p^2 + 5\right]\tanh^{sp}\theta\},\end{aligned} \tag{12}$$

$$\begin{aligned}\left[u^h\left(u^k\right)_x\right]_x = A^{h+k}\mu^2 ks\{&[s(h+k)-1]\tanh^{s(h+k)-2}\theta\\ &+[s(h+k)+1]\tanh^{s(h+k)+2}\theta\\ &-2s(h+k)\tanh^{s(h+k)}\theta\},\end{aligned} \tag{13}$$

$$\left(u^l\right)_x = A^l\mu sl\left\{\tanh^{sl-1}\theta - \tanh^{sl+1}\theta\right\}, \tag{14}$$

where

$$\theta = \mu(x-vt). \tag{15}$$

Substituting (7)–(15) into (6), we have

$$\begin{aligned}&lA^{l-1}\frac{\mathrm{d}A}{\mathrm{d}t}\tanh^{sl}\theta\\ &+A^l sl\left\{x\frac{\mathrm{d}\mu}{\mathrm{d}t} - \frac{\mathrm{d}(t\mu v)}{\mathrm{d}t}\right\}\left\{\tanh^{sl-1}\theta - \tanh^{sl+1}\theta\right\}\\ &+\lambda_1 A^m\mu sm\left\{\tanh^{sm-1}\theta - \tanh^{sm+1}\theta\right\}\\ &+\lambda_2 A^n\mu^3 sn\{(sn-1)(sn-2)\tanh^{sn-3}\theta\\ &-\left[2s^2n^2 + (sn-1)(sn-2)\right]\tanh^{sn-1}\theta\}\\ &+\lambda_2 A^n\mu^3 sn\{\left[2s^2n^2 + (sn+1)(sn+2)\right]\tanh^{sn+1}\theta\\ &-(sn+1)(sn+2)\tanh^{sn+3}\theta\}\\ &+\lambda_3 A^p\mu^4 sp\{(sp-1)(sp-2)(sp-3)\tanh^{sp-4}\theta\}\\ &+\lambda_3 A^p\mu^4 sp\{(sp+1)(sp+2)(sp+3)\tanh^{sp+4}\theta\\ &-4(sp-1)\left[s^2p^2 - 2sp + 2\right]\tanh^{sp-2}\theta\}\\ &+\lambda_3 A^p\mu^4 sp\{-4(sp+1)\left[s^2p^2 + 2sp + 2\right]\tanh^{sp+2}\theta\end{aligned}$$

$$+2sp\left[3s^2p^2+5\right]\tanh^{sp}\theta\big\}$$
$$+\lambda_4A^{h+k}\mu^2ks\big\{\left[s(h+k)-1\right]\tanh^{s(h+k)-2}\theta$$
$$+\left[s(h+k)+1\right]\tanh^{s(h+k)+2}\theta\big\}$$
$$+\lambda_4A^{h+k}\mu^2ks\big\{-2s(h+k)\tanh^{s(h+k)}\theta\big\}$$
$$+\lambda_5A^q\mu^2sq\big\{(sq-1)\tanh^{sq-2}\theta+(sq+1)\tanh^{sq+2}\theta$$
$$-2sq\tanh^{sq}\big\}+2\lambda_6A^l\tanh^{sl}\theta+[\lambda_7+\lambda_6x]A^l\mu sl$$
$$\cdot\big\{\tanh^{sl-1}\theta-\tanh^{sl+1}\theta\big\}=0. \quad (16)$$

Of course not all choices of the dependent exponents l,m,n,p,q, and r lead to the existence of an exact analytical soliton solution, since they must satisfy the homogeneous balance principle. This fact imposes some obvious restrictions on the values of these exponents in order to obtain a closed form solution that is physically meaningful. This may be a complicated task since we are concerned by a model equation with several dependent exponents.

As an example of the family of dark soliton solutions for the generalized KdV equation (6), we consider the resulting equation (16) at $q=h+k$. Under the later condition, (16) becomes

$$lA^{l-1}\frac{\mathrm{d}A}{\mathrm{d}t}\tanh^{sl}\theta$$
$$+A^lsl\left\{x\frac{\mathrm{d}\mu}{\mathrm{d}t}-\frac{\mathrm{d}(t\mu v)}{\mathrm{d}t}\right\}\big\{\tanh^{sl-1}\theta-\tanh^{sl+1}\theta\big\}$$
$$+\lambda_1A^m\mu sm\big\{\tanh^{sm-1}\theta-\tanh^{sm+1}\theta\big\}$$
$$+\lambda_2A^n\mu^3sn\big\{(sn-1)(sn-2)\tanh^{sn-3}\theta$$
$$-\left[2s^2n^2+(sn-1)(sn-2)\right]\tanh^{sn-1}\theta\big\}$$
$$+\lambda_2A^n\mu^3sn\big\{\left[2s^2n^2+(sn+1)(sn+2)\right]\tanh^{sn+1}\theta$$
$$-(sn+1)(sn+2)\tanh^{sn+3}\theta\big\}$$
$$+\lambda_3A^p\mu^4sp\big\{(sp-1)(sp-2)(sp-3)\tanh^{sp-4}\theta\big\}$$
$$+\lambda_3A^p\mu^4sp\big\{(sp+1)(sp+2)(sp+3)\tanh^{sp+4}\theta$$
$$-4(sp-1)\left[s^2p^2-2sp+2\right]\tanh^{sp-2}\theta\big\}$$
$$+\lambda_3A^p\mu^4sp\big\{-4(sp+1)\left[s^2p^2+2sp+2\right]\tanh^{sp+2}\theta$$
$$+2sp\left[3s^2p^2+5\right]\tanh^{sp}\theta\big\}$$
$$+\lambda_4A^q\mu^2ks\big\{(sq-1)\tanh^{sq-2}\theta+(sq+1)\tanh^{sq+2}\theta$$
$$-2sq\tanh^{sq}\theta\big\}+\lambda_5A^q\mu^2sq\big\{(sq-1)\tanh^{sq-2}\theta$$
$$+(sq+1)\tanh^{sq+2}\theta-2sq\tanh^{sq}\big\}+2\lambda_6A^l\tanh^{sl}\theta$$
$$+[\lambda_7+\lambda_6x]A^l\mu sl\big\{\tanh^{sl-1}\theta-\tanh^{sl+1}\theta\big\}=0. \quad (17)$$

By equating the exponents of the functions $\tanh^{sm+1}\theta$ and $\tanh^{sn+3}\theta$ in (17), one gets

$$sm+1=sn+3, \quad (18)$$

which yields the following analytical condition:

$$s=\frac{2}{m-n}. \quad (19)$$

It should be remarked that the dark soliton solution (7) can be obtained when $s>0$. Therefore the condition $m>n$ arises from (19).

Again, from (17), setting the coefficients of $\tanh^{sm+1}\theta$ and $\tanh^{sn+3}\theta$ to zero, we obtain

$$-\lambda_1A^m\mu sm-\lambda_2A^n\mu^3sn\,(sn+1)\,(sn+2)=0,$$

that gives

$$\mu^2=-\frac{\lambda_1mA^{m-n}}{\lambda_2n(sn+1)(sn+2)}. \quad (20)$$

Substituting (19) in (20) leads to

$$\mu=(m-n)\left[-\frac{\lambda_1A^{m-n}}{2\lambda_2n(m+n)}\right]^{\frac{1}{2}}, \quad (21)$$

which exists provided that $\lambda_1\lambda_2<0$.

Equating the exponents of $\tanh^{sq+2}\theta$ and $\tanh^{sp+4}\theta$ in (17) gives

$$sq+2=sp+4, \quad (22)$$

which in turn gives

$$s=\frac{2}{q-p} \quad (23)$$

for $q>p$. The same parametric condition (23) also results from matching up the exponents of $\tanh^{sq}\theta$ and $\tanh^{sp+2}\theta$.

Setting the coefficients of $\tanh^{sq+2}\theta$ and $\tanh^{sp+4}\theta$ terms to zero, we obtain

$$\begin{aligned}&+\lambda_3A^p\mu^4sp(sp+1)(sp+2)(sp+3)\\&+\lambda_4A^q\mu^2ks\,(sq+1)+\lambda_5A^q\mu^2sq(sq+1)=0,\end{aligned} \quad (24)$$

which gives

$$\mu=\left[\frac{(\lambda_5q+\lambda_4k)\,(sq+1)A^{q-p}}{\lambda_3p(sp+1)(sp+2)(sp+3)}\right]^{\frac{1}{2}}. \quad (25)$$

Substituting (23) into (25) gives

$$\mu = (q-p)\left[\frac{(\lambda_5 q+\lambda_4 k)A^{q-p}}{2\lambda_3 pq(p+q)}\right]^{\frac{1}{2}}, \tag{26}$$

which exists provided that $\lambda_3(\lambda_5 q+\lambda_4 k)>0$.

If we set the coefficients of $\tanh^{sq}\theta$ and $\tanh^{sp+2}\theta$ to zero in (17), we obtain a certain relation which also expresses the dependence of the inverse width μ on the coefficients λ_3, λ_4, and λ_5 such that $\lambda_3(\lambda_5 q+\lambda_4 k)>0$.

Equating the two values of s from (19) and (23) gives the condition

$$n+q=m+p \tag{27}$$

with $q>p$ and $m>n$. Thus the existence of the soliton solution (7) is the result of strict balance among third-order dispersion, dissipation, and diffusion, fourth-order nonlinear dispersion and nonlinearity effects described by (27).

Equating the two values of μ from (21) and (26) gives the parametric condition

$$\frac{\lambda_1\lambda_3}{\lambda_2(\lambda_5 q+\lambda_4 k)} = -\frac{n(m+n)}{pq(p+q)}. \tag{28}$$

Therefore, the condition (28) is crucial for the existence of dark solitons which are uniquely determined from the characteristics of the nonlinear medium, i.e. the model coefficients λ_i, $i=1-5$, and the dependent exponents n, m, p, q, and k.

Now, equating the exponents of $\tanh^{sl+1}\theta$ and $\tanh^{sn+1}\theta$ in (17) yields

$$l=n. \tag{29}$$

The coefficients of $\tanh^{sl}\theta$ terms in (17) gives

$$A(t) = A_0 l^{-1}\,\mathrm{e}^{-2\int_0^t \lambda_6 t'\,\mathrm{d}t'}, \tag{30}$$

where A_0 is an integral constant related to the initial pulse amplitude.

Also, the coefficients of $\tanh^{sn-3}\theta$ and $\tanh^{sn-1}\theta$ (with $l=n$) in (17), respectively yields

$$\lambda_2 A^n \mu^3 sn(sn-1)(sn-2)=0 \tag{31}$$

and

$$A^l sl\left\{x\left[\frac{\mathrm{d}\mu}{\mathrm{d}t}+\lambda_6\mu\right]-\mu v - t\frac{\mathrm{d}(\mu v)}{\mathrm{d}t} - \lambda_2\mu^3\left[2-3sn+3s^2n^2\right]+\lambda_7\mu\right\}=0. \tag{32}$$

Finally, the coefficients of $\tanh^{sl+1}\theta$ and $\tanh^{sn+1}\theta$ terms in (17) gives

$$A^l sl\left\{-x\left[\frac{\mathrm{d}\mu}{\mathrm{d}t}+\lambda_6\mu\right]+\mu v + t\frac{\mathrm{d}(\mu v)}{\mathrm{d}t} + \lambda_2\mu^3\left[2+3sn+3s^2n^2\right]+3\lambda_1 A^{2n}\mu-\lambda_7\mu\right\}=0. \tag{33}$$

To solve (31)–(33), we consider the two cases:

3.1. Case 1: $sn-1=0$

In this case we find

$$s=\frac{1}{n}. \tag{34}$$

Substituting (34) into (19), (21), and (32), respectively, gives

$$m=3n, \tag{35}$$

$$\mu^2 = -\frac{A^{2n}\lambda_1(t)}{3n\lambda_2(t)}, \tag{36}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}+\lambda_6(t)\mu=0, \tag{37}$$

$$\frac{\mathrm{d}(t\mu v)}{\mathrm{d}t} = \lambda_7(t)\mu - 2\lambda_2(t)\mu^3. \tag{38}$$

Note that (37) and (38) are obtained from the fact that the parameter v we want to determine through (32) is a function of time, then we have split (32) into the two equations (37) and (38).

Integrating (37) and (38), we get

$$\mu(t) = \mu_0\,\mathrm{e}^{-\int_0^t \lambda_6(t')\,\mathrm{d}t'}, \tag{39}$$

$$v(t) = \frac{1}{t\mu(t)}\int_0^t \left[\lambda_7(t')\mu(t') - 2\lambda_2(t')\mu^3(t')\right]\mathrm{d}t'. \tag{40}$$

Inserting the expressions (30) and (39) into (36), we obtain the following constraint among $\lambda_1(t), \lambda_2(t)$, and $\lambda_6(t)$:

$$\frac{\lambda_1(t)}{\lambda_2(t)} = -\frac{n^{-2n}A_0^{2n}\,\mathrm{e}^{(2-4n)\int_0^t \lambda_6(t')\,\mathrm{d}t'}}{2\mu_0^2} \tag{41}$$

with $\lambda_1(t)\lambda_2(t)<0$.

3.2. Case 2: $sn-2=0$

In this case we find

$$s=\frac{2}{n}. \tag{42}$$

By substituting (42) into (19), (21), and (32), respectively, we obtain

$$m=2n, \tag{43}$$

$$\mu^2=-\frac{\lambda_1(t)A^n}{6\lambda_2(t)}, \tag{44}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}+\lambda_6(t)\mu=0, \tag{45}$$

$$\frac{\mathrm{d}(t\mu v)}{\mathrm{d}t}=\lambda_7(t)\mu-8\lambda_2(t)\mu^3. \tag{46}$$

Integrating (45) and (46) gives

$$\mu(t)=\mu_0\,\mathrm{e}^{-\int_0^t\lambda_6(t')\,\mathrm{d}t'}, \tag{47}$$

$$v(t)=\frac{1}{t\mu(t)}\int_0^t\left[\lambda_7(t')\mu(t')-8\lambda_2(t')\mu^3(t')\right]\mathrm{d}t'. \tag{48}$$

Substituting (30) and (47) into (44), we obtain the following constraint among $\lambda_1(t),\lambda_2(t)$, and $\lambda_6(t)$:

$$\frac{\lambda_1(t)}{\lambda_2(t)}=-\frac{n^{-n}A_0^n\,\mathrm{e}^{2(1-n)\int_0^t\lambda_6(t')\,\mathrm{d}t'}}{6\mu_0^2} \tag{49}$$

with $\lambda_1(t)\lambda_2(t)<0$.

Lastly, we can determine the dark soliton solutions for the t-dependent KdV equation (6) when we substitute (30), (34), (36), (39), (40) in (7) with the respective constraint (41) for the first case of solution or we substitute (30), (42), (43), (47), (48) in (7) with the respective constraint (49) for the second case of solution as

$$u(x,t)=A\tanh^p\{\mu(x-vt)\}, \tag{50}$$

which exists provided that $m>n$, $q>p$, $n+q=m+p$, $l=n$ with the condition (28).

4. Conclusion

In this paper, we have studied the dynamics of dark solitons within the framework of a family of fully nonlinear perturbed KdV equations with time varying coefficients. Besides the pure KdV equation, the additional terms that are taken into account are nonlinearity, fourth-order dispersion, diffusion, dissipation, and reaction. A solitary wave ansatz has been used to carry out the integration and an exact dark soliton solution is obtained. All the physical parameters in the soliton solution are obtained as function of the dependent model coefficients and exponents. The conditions of existence of solitons are presented. We hope that this paper will help to understand the behaviour of solitons in very complicated KdV systems.

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